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NECESSARY AND SUFFICIENT CRITERIA FOR ELLIPTICITY OF THE EQUILIBRIUM EQUATIONS OF A NON-LINEARLY ELASTIC MEDIUM†

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Necessary and sufficient conditions are found for ellipticity of the equilibrium equations of a homogeneous isotropic compressible elastic material. These conditions comprise a finite system of elementary inequalities imposing explicit constraints on the strain energy density of the material and the principal relative elongations, as well as a series of relationships for domains in which certain polynomials of one real variable, whose coefficients are determined by the energy density function and the principal elongations, remain constant in sign. The degrees of the polynomials are one, two and six, respectively. An effective sufficient condition is formulated that guarantees ellipticity of the equilibrium equations and does not contain any auxiliary parameters. It is shown that if the material satisfies certain physically plausible and not overly restrictive conditions, the ellipticity criterion admits of a simpler formulation, obviating the need to investigate polynomials.

The ellipticity condition for the equilibrium equations of a non-linearly elastic medium [1–3], which excludes the existence of weak discontinuity surfaces of the field of displacements, imposes certain restrictions on the strain energy density of an elastic material Π and may be considered one of the constitutive inequalities of the non-linear theory of elasticity [1, 2]. An alternative viewpoint associates the breakdown of ellipticity in the equilibrium equations with certain effects involving loss of stability in elastic bodies [4–7]. Each of these approaches indicates the significance of the ellipticity concept in the theory of elasticity.

Before giving rigorous definitions, we will introduce some notation. Let w and W be the domains occupied by an elastic body before and after deformation, respectively, and $\mathbf{r}$ and $\mathbf{R}$ the position vectors of an arbitrary particle in configurations w and W, respectively. A deformation of the body will be any continuously differentiable and bijective transformation $\mathbf{R} = f(\mathbf{r})$ of w onto the domain $W = f(w)$ which preserves the orientation at each point $\mathbf{r} \in w$. It is assumed that the material is compressible and hyperelastic [1, 2], that is, it possesses a potential energy function $\Pi = \Pi(\mathbf{r}, \mathbf{C})$ where $\mathbf{C} = \nabla\mathbf{R}$ is the deformation gradient. Let $\mathbf{e}_m$ $(m = 1, 2, 3)$ denote the unit vectors of some fixed Cartesian system of coordinates. Then the equilibrium equations of the elastic body may be written as follows [3]:

$$\frac{\partial^2\Pi(\mathbf{r}, \mathbf{C})}{\partial C_{ij}\partial C_{kl}} \frac{\partial^2 X_l(r)}{\partial x_i \partial x_k} + \frac{\partial^2\Pi(\mathbf{r}, \mathbf{C})}{\partial x_i \partial C_{ij}} + \rho_0(\mathbf{r}) g_j(\mathbf{r}, \mathbf{C}) = 0 \quad (j = 1,2,3) \tag{0.1}$$

where x_k, X_l, $C_{kl} = \partial X_l/\partial x_k$, g_k $(k, l = 1, 2, 3)$ are the components of the quantities relative to the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{g} = \mathbf{g}(\mathbf{r}, \mathbf{C})$ is the density of body forces, and $\rho_0 = \rho_0(\mathbf{r})$ is the density of the material in the reference configuration w. Summation from 1 to 3 is assumed over repeated indices. The second derivatives of the potential $\Pi(\mathbf{r}, \mathbf{C})$ which occur in Eq. (0.1), as well as the quantities $\rho_0(\mathbf{r})$ and $\mathbf{g}(\mathbf{r}, \mathbf{C})$, are assumed to be continuous in their domains of definition. For the density $\rho_0(\mathbf{r})$ this domain is the set is the set w, and for the density $\mathbf{g}(\mathbf{r}, \mathbf{C})$ it is the set $w \times D$, where D is the space of non-singular tensors of rank two over Euclidean three-space, equipped with the Euclidean norm [1, 2].

System (0.1) is quasilinear, since it is linear in the highest-order derivatives of the unknown functions $X_l(\mathbf{r})$ $(l = 1, 2, 3)$. The fourth-rank tensor $\mathbf{A} = \mathbf{A}(\mathbf{r}, \mathbf{C})$ with components $A_{ijkl} = \partial^2\Pi/\partial C_{ij}\partial C_{kl}$ relative to the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is known as the elasticity tensor [1, 2].

Definition 1. A statically possible deformation of the elastic body described above is any twice differentiable and bijective mapping $\mathbf{R} = f(\mathbf{r})$ of the set w onto the set $W = f(w)$ which satisfies the equilibrium equations (0.1) and preserves orientation at each point $\mathbf{r} \in w$.

Definition 2. A quasi-linear system (0.1) is said to be elliptic [2] (or ordinarily elliptic) at a point $\mathbf{r}_0 \in w$ for a given statically possible deformation $\mathbf{R}_* = f_*(\mathbf{r})$ if, for any unit vector $\mathbf{a}$

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$$\det\|A_{ijkl}[\mathbf{r}_0, \mathbf{C}_*(\mathbf{r}_0)]a_i a_k\|_{j,l=1,2,3} \neq 0 \quad (\mathbf{a}\cdot\mathbf{a}=1) \tag{0.2}$$

$$\mathbf{C}_* = \nabla\mathbf{R}_* = \nabla f_*(\mathbf{r}), \quad a_m = \mathbf{a}\cdot\mathbf{e}_m \quad (m=1,2,3)$$

If condition (0.2) is satisfied at each point $\mathbf{r}_0 \in w$, system (0.1) is said to be elliptic for the statically possible deformation $\mathbf{R}_* = f_*(\mathbf{r})$. Finally, system (0.1) is said to be elliptic in w if it has this property for any statically possible deformation of the body.

Definition 3. A quasilinear system (0.1) satisfies Hadamard's inequality [1, 2] (or is weakly elliptic) at a point $\mathbf{r}_0 \in w$ for a given statically possible deformation $\mathbf{R}_* = f_*(\mathbf{r})$ if, for any pair of unit vectors **a**, **b**

$$A_{ijkl}[\mathbf{r}_0, \mathbf{C}_*(\mathbf{r}_0)]a_i a_k b_j b_l \geqslant 0 \quad (\mathbf{a}\cdot\mathbf{a}=\mathbf{b}\cdot\mathbf{b}=1) \tag{0.3}$$

If condition (0.3) is satisfied at each point $\mathbf{r}_0 \in w$, we shall say that the system satisfies Hadamard's inequality for the deformation $\mathbf{R}_* = f_*(\mathbf{r})$. Finally, a quasilinear system (0.1) satisfies Hadamard's inequality in the domain w if it has this property for any statically possible deformation of the body.

The properties of strong ellipticity and positive longitudinal elasticity [1, 2] are defined in similar terms; the respective conditions on the tensor **A** are

$$A_{ijkl}[\mathbf{r}_0, \mathbf{C}_*(\mathbf{r}_0)]a_i a_k b_j b_l > 0 \quad (\mathbf{a}\cdot\mathbf{a}=\mathbf{b}\cdot\mathbf{b}=1) \tag{0.4}$$

$$A_{ijkl}[\mathbf{r}_0, \mathbf{C}_*(\mathbf{r}_0)]a_i a_k a_j a_l > 0 \quad (\mathbf{a}\cdot\mathbf{a}=1) \tag{0.5}$$

It follows from (0.2)–(0.5) that strong ellipticity implies Hadamard's inequality, ordinary ellipticity and positive longitudinal elasticity.

The physical meaning of these ideas is as follows. Hadamard's inequality (0.3) is a necessary condition for an arbitrary statically possible deformation of an elastic body to be stable with respect to "dead" external forces, and also under mixed boundary conditions, when displacements are prescribed on part of the body's surface and a "dead" load on the remainder. In addition, it guarantees that the velocities of propagation of acceleration waves in an elastic medium are real [1, 2].

Strong ellipticity (0.4) guarantees stability of a homogeneous deformation of an elastic body with a rigidly attached boundary surface; it also implies that the squares of the velocities of propagation of acceleration waves in an elastic medium are positive [1, 2]. It is also a necessary condition for existence "in the small", i.e. in a brief time interval, of a solution of the first boundary-value problem (with displacements prescribed on the boundary) of the dynamic equations of the non-linear theory of elasticity [8]. In addition, if the external forces applied to the body are conservative, when the corresponding boundary-value problem for the equilibrium equations (0.1) reduce to the problem of determining when the potential energy functional of the body is stationary, the strong ellipticity condition plays an important role in investigating whether the functions that reach a stationary value are regular or not [9].

Positive longitudinal elasticity (0.5) is a necessary condition for the existence of at least one longitudinal acceleration wave in an elastic body [1]. At first glance, this statement might appear trivial; but one must bear in mind that in a compressible, non-linearly elastic medium, unlike the situation in linearly elastic media, an acceleration wave is in general neither longitudinal nor transverse.

Finally, a most important corollary of ordinary ellipticity of Eqs (0.1) in a domain w is that any statically possible deformation of the body is a twice continuously differentiable mapping of w onto a domain D. In other words, if system (0.1) is elliptic in w, then in the equilibrium state of the body the displacement field cannot have any weak discontinuity surfaces. Moreover, it has been proved [10] that for an isotropic incompressible material satisfying the Baker–Ericksen inequalities [1, 2] (see below, (2.17)), ellipticity is equivalent to strong ellipticity, so that ellipticity guarantees the existence of all the effects just associated with strong ellipticity.

It should be mentioned that the results of this paper will imply that, for an isotropic compressible elastic material satisfying the Baker–Ericksen inequalities and the TE^+-conditions [1], ellipticity ensures that the squares of the velocities of propagation of weak discontinuity waves are positive, at least for all directions N that lie in the principal planes of the Finger strain tensor $\mathbf{F} = \mathbf{C}^T \cdot \mathbf{C}$ [1, 2] (the superscript T indicates transposition of second-rank tensors). But if the tensor **F** is transtropic or isotropic at a point $\mathbf{r}_0$ of the body, the squares of the velocities of propagation of the acceleration waves are positive at $\mathbf{r}_0$ for any wave normal **N**.

Note that under the previous assumptions as to the regularity of the quantities $\rho_0(\mathbf{r})$, $\mathbf{g}(\mathbf{r}, \mathbf{C})$, $\Pi(\mathbf{r}, \mathbf{C})$ it can be shown that, for any compressible hyperelastic material (including inhomogeneous and anisotropic materials), the properties of ordinary and weak ellipticity together are equivalent to strong ellipticity.

Since the ideas described above are defined in terms of the strain energy density $\Pi(\mathbf{r}, \mathbf{c})$ of the material, one can associate them directly with the material, speaking, say, of the latter's weak or strong ellipticity. The definitions adopted here can also be generalized to more complicated (not necessarily quasilinear) non-linear systems.

The main purpose of this paper is to establish new relations among the properties of ordinary, weak and strong ellipticity (as well as further inequalities of the non-linear theory of elasticity), and also to simplify the ordinary ellipticity criterion (0.2), reducing it to a form more convenient for practical purposes. Effective ways of verifying conditions (0.3) and (0.4) for isotropic elastic materials are already known (see below, (2.8) and (2.10), for the relevant references). The same is true of positive longitudinal elasticity [11].

Conditions (0.2)–(0.5) are not infrequently formulated in terms of the acoustic tensor of the elastic medium, $\mathbf{Q} = \mathbf{Q}(\mathbf{N})$ [1–3], which is related to the elasticity tensor $\mathbf{A}$ by the formula

$$\mathbf{Q} = \mathbf{N} \cdot \mathbf{C}^T \cdot \mathbf{A}^{T(3,4)} \cdot \mathbf{C} \cdot \mathbf{N} \tag{0.6}$$

where $\mathbf{N}$ is an arbitrary unit vector. Apart from $\mathbf{N}$, $\mathbf{Q}$ also depends on $\mathbf{r}$ and $\mathbf{C}$, though these arguments have been omitted for brevity. The label $T(3, 4)$ in (0.6) denotes transposition of the fourth-rank tensor $\mathbf{A}$ with respect to the third and fourth indices. We know [1, 2] that the eigenvalues Q_m ($m = 1, 2, 3$) of $\mathbf{Q}(\mathbf{N})$, called the acoustic numbers, can be expressed in terms of the velocities s_m ($m = 1, 2, 3$) of propagation of acceleration waves with wave normal $\mathbf{N}$; the formula is $Q_m = \rho s_m^2$, where ρ is the density of the medium in the actual configuration. Since $\mathbf{C}$ is a non-singular tensor, it is clear that Hadamard's inequality (0.3) (the strong ellipticity condition (0.4)) is equivalent to the condition that the tensor $\mathbf{Q}(\mathbf{N})$ be positive semi-definite (positive definite) for all directions $\mathbf{N}$; while the condition of positive longitudinal elasticity (0.5) reduces to the requirement that $\mathbf{N} \cdot \mathbf{Q}(\mathbf{N}) \cdot \mathbf{N} > 0$ ($\forall\, \mathbf{N} : \mathbf{N} \cdot \mathbf{N} = 1$).

1. In terms of the acoustic tensor $\mathbf{Q}(\mathbf{N})$, the ellipticity condition for the equilibrium equations (0.1) can be written as follows [3]:

$$\det \mathbf{Q}(\mathbf{N}) \neq 0 \quad (\forall \mathbf{N} : \mathbf{N} \cdot \mathbf{N} = 1) \tag{1.1}$$

Since this relationship involves three arbitrary parameters (the components of the normal $\mathbf{N}$), it is usually quite difficult to verify inequality (1.1) for actual materials. It is therefore important to derive conditions, directly involving the potential Π and deformation gradient $\mathbf{C}$ (at a fixed point $\mathbf{r}_0$ of w), that do not contain any subsidiary parameters and guarantee the validity of condition (1.1) for any $\mathbf{N}$ ($\mathbf{N} \cdot \mathbf{N} = 1$). This problem was solved for a homogeneous, isotropic, incompressible material in [10], where elimination of the vector $\mathbf{N}$ led to a system of elementary inequalities equivalent to the ellipticity condition and imposing explicit restrictions on the principal extensions υ_q ($q = 1, 2, 3$) [1, 2] and the first and second derivatives of the potential Π with respect to them. Necessary and sufficient conditions have been found for ellipticity of the equilibrium equations of the plane theory of elasticity in the case of compressible [12] and incompressible [13] materials.

Analogous conditions will be derived below for the three-dimensional equilibrium equations (0.1) of a compressible, non-linearly elastic medium, on the assumption that it is homogeneous and isotropic. In that case the acoustic tensor $\mathbf{Q}(\mathbf{N})$ does not depend explicitly on the position vector $\mathbf{r}$, while the principle of material objectivity [1, 2] reduces the dependence on the deformation gradient $\mathbf{C}$ to dependence on the Finger strain tensor $\mathbf{F} = \mathbf{C}^T \cdot \mathbf{C}$. In addition, the potential Π is a function of the principal stretches only.

We will use the known representation of the components of the acoustic tensor $\mathbf{Q}(\mathbf{N})$ relative to the principal axes of the tensor $\mathbf{F}$ [1–3] (i, j, k stands for an arbitrary permutation of the indices 1, 2, 3)

$$JQ_{ij} = \gamma_k M_i M_j, \quad M_q = \upsilon_q N_q \tag{1.2}$$

$$JQ_{kk} = \alpha_j M_i^2 + \alpha_i M_j^2 + \beta_k M_k^2, \quad J = \det \mathbf{C}$$

$$\alpha_k = (\Pi_i \upsilon_i - \Pi_j \upsilon_j)/(\upsilon_i^2 - \upsilon_j^2), \quad \beta_k = \Pi_{kk} \tag{1.3}$$

$$\gamma_k = (\gamma_k^+ - \gamma_k^-)/2, \quad \gamma_k^\pm = \pm \Pi_{ij} + (\Pi_i \mp \Pi_j)/(\upsilon_i \mp \upsilon_j)$$

$$\Pi_m \equiv \partial \Pi / \partial \upsilon_m, \quad \Pi_{mn} \equiv \partial^2 \Pi / \partial \upsilon_m \partial \upsilon_n \quad (m, n = 1, 2, 3)$$

where N_q are the components of $\mathbf{N}$ relative to an orthonormal basis Σ consisting of eigenvectors of the Finger tensor. Throughout this paper, $q = 1, 2, 3$. Henceforth, unless otherwise stated, all components of non-scalar quantities will be taken relative to the basis Σ. The function $\Pi(\upsilon_1, \upsilon_2, \upsilon_3)$ is assumed to be twice continuously differentiable in its domain of definition ($\upsilon_1 > 0, \upsilon_2 > 0, \upsilon_3 > 0$) and symmetric (i.e. invariant with respect to any permutation of its arguments).

The above symmetry is implied by the assumption that the material is isotropic. Thus, the quantities α_q, β_q, γ_q, $\gamma_q^\pm$, expressed in terms of the potential Π and principal extensions υ_q, exist and are continuous for any positive values of υ_1, υ_2 and υ_3. These quantities play an auxiliary role and, with the exception of α_q and β_q, do not admit of an immediate mechanical interpretation. As to α_q and β_q, they are proportional to the squares of the velocities of propagation of the principal acceleration waves [1, 2] in a homogeneously deformed elastic medium with principal extensions υ_q. To be precise: if i, j, k is any permutation of the numbers 1, 2, 3 and we let s_{kk} denote the velocity of the principal longitudinal wave, which propagates along the kth principal direction of the Finger tensor, and let s_{ij} denote the velocity of the principal transverse wave, which propagates in the ith and is polarized along the jth principal directions, then

$$\alpha_k = \frac{\rho_0 s_{ij}^2}{\upsilon_i^2} = \frac{\rho_0 s_{ji}^2}{\upsilon_j^2}, \quad \beta_k = \frac{\rho_0 s_{kk}^2}{\upsilon_k}$$

It turns out that in real elastic materials the quantities α_k, β_k, γ_q, $\gamma_q^{\pm}$ may be determined empirically, through purely static experiments. In fact, it follows from formulae (1.3) and the law of state of an isotropic elastic body in Finger's form [1, 2] that

$$\alpha_k = J(t_i - t_j)/(\upsilon_i^2 - \upsilon_j^2), \quad \beta_k = J\upsilon_k^{-1} t_{k,k}$$

$$2\gamma_k = \upsilon_k[(t_i - t_j)(\upsilon_i^2 + \upsilon_j^2)/(\upsilon_i^2 - \upsilon_j^2) + (t_{i,j}\upsilon_j + t_{j,i}\upsilon_i)]$$

$$2\gamma_k^{\pm} = \pm\upsilon_k[(t_i - t_j)(\upsilon_i \pm \upsilon_j)/(\upsilon_i \mp \upsilon_j) + (t_{i,j}\upsilon_j + t_{j,i}\upsilon_i)]$$

$$t_{m,n} \equiv \partial t_m / \partial \upsilon_n \quad (m,n = 1,2,3), \quad J = \upsilon_1 \upsilon_2 \upsilon_3$$

where t_q are the principal Cauchy stresses, which is an isotropic material are functions of the principal extensions υ_q. Note that the squares of the principal extensions υ_q are the eigenvalues of the tensor $\mathbf{F}$, while the numbers $\upsilon_q - 1$ are the principal relative elongations.

Using (1.2), we can write

$$J^3 \det \mathbf{Q} = \Psi(\mathbf{m}) \equiv \delta_1 m_1^3 + \delta_2 m_2^3 + \delta_3 m_3^3 + (\varepsilon_{12} m_1 + \varepsilon_{21} m_2) m_3^2 + (\varepsilon_{23} m_2 + \varepsilon_{32} m_3) m_1^2 + \\ +(\varepsilon_{31} m_3 + \varepsilon_{13} m_1) m_2^2 + \theta m_1 m_2 m_3 \tag{1.4}$$

$$m_q \equiv M_q^2, \quad \delta_k = \beta_k \alpha_i a_j$$

$$\kappa_k = \beta_i \beta_j + \alpha_k^2 - \gamma_k^2, \quad \varepsilon_{ij} = \alpha_i \kappa_j + \alpha_j \alpha_k \beta_k \tag{1.5}$$

$$\theta = \beta_1 \beta_2 \beta_3 + 2(\alpha_1 \alpha_2 \alpha_3 + \gamma_1 \gamma_2 \gamma_3) + \sum_{q=1}^{3} \beta_q \gamma_q^+ \gamma_q^-$$

where $\mathbf{m}$ is the vector with components m_q relative to the basis Σ.

Lemma 1. Inequality (1.1) is valid if and only if the form $\Psi(\mathbf{m})$ is either positive for all non-zero $\mathbf{m}$ or negative for all of them.

Proof. Indeed, otherwise, since $\Psi(\mathbf{m})$ is a continuous function, there must be a value $\mathbf{m}^0$ of $\mathbf{m}$ such that $\Psi(\mathbf{m}^0) = 0$; but by (1.4) this violates condition (1.1).

Before formulating the main theorem, we introduce some notation

$$A_{ki}(t) = \varepsilon_{ij} t + \varepsilon_{ji}, \quad B_{ki}(t) = \varepsilon_{kj} t^2 + \theta t + \varepsilon_{ki}$$

$$C_{ki}(t) = \delta_i t^3 + \varepsilon_{jk} t^2 + \varepsilon_{ik} t + \delta_j \tag{1.6}$$

$$D_{ki}(t) = 4[\delta_k B_{ki}^3(t) + C_{ki}(t) A_{ki}^3(t)] + 108\delta_k^2 C_{ki}^2(t) - [A_{ki}(t) B_{ki}(t) + 9\delta_k C_{ki}(t)]^2$$

$$R^+ = \{t \in R : t > 0\}, \quad X_{ki} = \{t \in R^+ : D_{ki}(t) > 0\} \tag{1.7}$$

$$Y_{ki} = \{t \in R^+ : A_{ki}(t) B_{ki}(t) > 0\}$$

where R is the set of real numbers and t is a real variable. Let V denote the space of principal extensions, i.e. the collection of points $\upsilon \equiv (\upsilon_1, \upsilon_2, \upsilon_3)$ of real three-space R, with positive components $(\upsilon_1, \upsilon_2, \upsilon_3)$.

Theorem 1. A homogeneous isotropic compressible elastic material has the ellipticity property at a given point $\upsilon \in V$ if and only if, for any permutation i, j, k of the indices 1, 2, 3, the following conditions are satisfied

1. the following inequalities hold

$$\alpha_i\alpha_j > 0,\quad \beta_i\beta_j > 0$$
$$\left[\gamma_k^+ + \sqrt{\beta_i\beta_j}\,\mathrm{sign}(\beta_i+\beta_j)\right]\left[\gamma_k^- + \sqrt{\beta_i\beta_j}\,\mathrm{sign}(\beta_i+\beta_j)\right]\mathrm{sign}[\alpha_k(\beta_i+\beta_j)] > 0 \tag{1.8}$$

2. the sets X_{ki}, Y_{ki} form a cover of the positive semi-axis R^+, i.e.

$$X_{ki} \cup Y_{ki} = R^+ \tag{1.9}$$

Proof. To prove the necessity of inequalities (1.8), consider the restriction $\Psi_k(m_i, m_j)$ of the function $\Psi(\mathbf{m})$ to the coordinate plane $m_k = 0$

$$\Psi_k(m_i,m_j) = (\alpha_j m_i + \alpha_i m_j)(\alpha_k\beta_i m_i^2 + \alpha_k\beta_j m_j^2 + \kappa_k m_i m_j) \tag{1.10}$$

Each of the factors on the right of (1.10) must be non-zero for all m_i, m_j such that $m_i \geqslant 0$, $m_j \geqslant 0$, $m_i + m_j > 0$. By virtue of the notation (1.3), (1.5) this immediately implies inequality (1.8).

Note that conditions (1.8) are not only necessary but also sufficient for (1.1) to hold in the coordinate planes $m_q = 0$. Henceforth, therefore, we may assume that $m_q > 0$.

We will now write expression (1.4) for $\Psi(\mathbf{m})$ as an expansion in powers of m_k

$$\Psi(\mathbf{m}) = \delta_k m_k^3 + a_k(m_i,m_j)m_k^2 + b_k(m_i,m_j)m_k + c_k(m_i,m_j) \tag{1.11}$$
$$a_k(m_i,m_j) = \varepsilon_{ij}m_i + \varepsilon_{ji}m_j$$
$$b_k(m_i,m_j) = \varepsilon_{kj}m_i^2 + \theta m_i m_j + \varepsilon_{ki}m_j^2$$
$$c_k(m_i,m_j) = \delta_i m_i^3 + \varepsilon_{jk}m_i^2 m_j + \varepsilon_{ik}m_i m_j^2 + \delta_j m_j^3$$

For fixed m_i and m_j, the right-hand side of (1.11) is a cubic polynomial in m_k. Suppose first that $\beta_k > 0$. Then, be (1.8) and (1.5), we have $\delta_k > 0$, and by Lemma 1 it follows that for all non-zero $\mathbf{m}$ necessarily $\Psi(\mathbf{m}) > 0$.

The necessary and sufficient conditions obtained in [10] for the cubic polynomial

$$f(t) = \alpha t^3 + \beta t^2 + \gamma t + \delta \quad (\alpha > 0,\ \delta > 0)$$

to be positive for all $t > 0$ may be written in the form

$$(\varepsilon > 0) \vee [(\beta > 0) \wedge (\gamma > 0)] \tag{1.12}$$
$$\varepsilon \equiv 4(\alpha\gamma^3 + \delta\beta^3) + 108\alpha^2\delta^2 - (\beta\gamma + 9\alpha\delta)^2$$

Applied to the polynomial (1.11), this condition becomes

$$[d_k(m_i,m_j) > 0] \vee \{[a_k(m_i,m_j) > 0] \wedge [b_k(m_i,m_j) > 0]\} \tag{1.13}$$
$$d_k(m_i,m_j) \equiv 4[\delta_k b_k^3(m_i,m_j) + c_k(m_i,m_j)a_k^3(m_i,m_j)] +$$
$$+108\delta_k^2 c_k^2(m_i,m_j) - [a_k(m_i,m_j)b_k(m_i,m_j) + 9\delta_k c_k(m_i,m_j)]^2$$

Note that the condition $c_k(m_i, m_j) > 0$ is satisfied for all admissible values of m_i and m_j, because of (1.8) and the obvious equality $c_k(m_i, m_j) = \Psi_k(m_i, m_j)$.

Since $m_j \neq 0$, we may define $t = m_i/m_j$. Then, in the notation of (1.6), condition (1.13) is equivalent to

$$[D_{ki}(t) > 0] \vee \{[A_{ki}(t) > 0] \wedge [B_{ki}(t) > 0]\} \tag{1.14}$$

The case $\beta_k < 0$ is treated similarly. Here (1.14) is replaced by the condition

$$[D_{ki}(t) > 0] \vee \{[A_{ki}(t) < 0] \wedge [B_{ki}(t) < 0]\} \tag{1.15}$$

Combining conditions (1.14) and (1.15), we obtain

$$[D_{ki}(t)>0]\vee[A_{ki}(t)B_{ki}(t)>0] \tag{1.16}$$

The sign of β_k plays no part in condition (1.16). Since the condition must hold for any $t > 0$, we conclude, in view of the notation (1.7), that (1.9) is valid.

A close analysis of the arguments up to this point will show that conditions (1.8) and (1.9) are not only necessary but also sufficient. The theorem is proved.

Remarks to Theorem 1. 1. Inequalities (1.8) indicate that if the ellipticity condition holds, the parameters α_q are either all positive or all negative. The same is true of the parameters β_k. From the physical point of view this means that the acoustic numbers for the principal transverse amplitudes are either all positive or all negative. The same is true for the acoustic numbers corresponding to the principal longitudinal amplitudes. Thus, ellipticity either entirely excludes the existence of principal waves or admits of the existence of all nine principal waves, of only three longitudinal waves or of only six transverse waves. No other version is consistent with ellipticity. By "principal" waves we mean here waves whose wave normals are directed along the principal axes of the Finger strain tensor.

2. If inequalities (1.8) are valid for any permutation i, j, k of the numbers 1, 2, 3, it follows from the proof of the theorem that relations (1.9), considered for all possible permutations of the indices, are either all true or all false. Hence it follows that only one of the six equalities (1.9) is independent. Therefore, the ellipticity condition for a homogeneous isotropic compressible material is equivalent to a system of nine elementary inequalities (1.8) and one set-theoretical relation of type (1.9). This fact may be used to reduce the amount of computation to check for their correctness.

Thus, in order to determine whether a given material is or is not elliptic, one has to analyse a finite system of elementary inequalities (1.8) and to determine the domains in which the polynomials $D_{ki}(t)$, $A_{ki}(t)$, $B_{ki}(t)$ remain fixed in sign. For the last two, which are respectively linear and quadratic, the problem involves no difficulties and can be solved by analytical means. The polynomial $D_{ki}(t)$, however, is of degree six, so that, in general, determination of the domain X_{ki} requires the use of computers.

As an example, let us consider a Blatz–Ko material [2, 4] with potential

$$\Pi=\frac{1}{2}\mu\left[I_1+\frac{1}{\nu}(I_3^{-\nu}-1)-3\right]\quad\left(\mu>0,\quad \nu\geqslant-\frac{1}{2}\right) \tag{1.17}$$

where μ and ν are constants (in particular, μ is the shear modulus of the material for small deformations from the natural state), and I_1 and I_3 are the first and third principal invariants of the Finger strain tensor, which are related to the principal extensions by the relations

$$I_1=\upsilon_1^2+\upsilon_2^2+\upsilon_3^2,\quad I_3=\upsilon_1^2\upsilon_2^2\upsilon_3^2 \tag{1.18}$$

Using formulae (1.3), (1.5), (1.17) and (1.18), we obtain

$$\begin{gathered}\alpha_k=\mu,\quad \beta_k=\mu[1+(2\nu+1)\upsilon_k^{-2}I_3^{-\nu}]\\ \gamma_k^{\pm}=\mu[1\pm(2\nu+1)\upsilon_i^{-1}\upsilon_j^{-1}I_3^{-\nu}]\\ \kappa_k=\mu(\beta_i+\beta_j),\quad \gamma_k=\mu(2\nu+1)\upsilon_i^{-1}\upsilon_j^{-1}I_3^{-\nu}\\ \varepsilon_{ij}=\mu^2(\beta_i+2\beta_k),\quad \theta=2\mu^2(\beta_1+\beta_2+\beta_3)\end{gathered} \tag{1.19}$$

Using formulae (1.19), one can verify inequalities (1.8) for any deformation. In addition, since in this case the parameters θ and ε_{pq} (p, q = 1, 2, 3) are positive for all admissible values of υ_1, υ_2, υ_3 it follows from (1.6) and (1.7) that $Y_{ki} = R^+$, that is, equalities (1.9) are true. Thus, it follows from Theorem 1 that the material (1.17) is elliptic at each point $\upsilon \in V$. This could have been expected since, as shown in [2], this material is strongly elliptic for any deformation.

Theorem 1 yields a sufficient condition for ellipticity of the equilibrium equations. We will first introduce some notation

$$\lambda_k=\delta_i\varepsilon_{ik}^3+\delta_j\varepsilon_{jk}^3,\quad \mu_{ki}=3\delta_k\theta\varepsilon_{ki}^2+\varepsilon_{ji}^2(3\delta_j\varepsilon_{ij}+\varepsilon_{ik}\varepsilon_{ji})$$

$$\nu_{ki} = \zeta_{ki} + \varepsilon_{ji}(\varepsilon_{jk}\varepsilon_{ji}^2 + 3\varepsilon_{ik}\varepsilon_{ij}\varepsilon_{ji} + 3\delta_j\varepsilon_{ij}^2)$$

$$\pi_k = \chi_k + \delta_i\varepsilon_{ji}^3 + 3\varepsilon_{jk}\varepsilon_{ij}\varepsilon_{ji}^2 + 3\varepsilon_{ik}\varepsilon_{ij}^2\varepsilon_{ji} + \delta_j\varepsilon_{ij}^3$$

$$\zeta_{ki} = 3\delta_k\varepsilon_{ki}(\theta^2 + \varepsilon_{ki}\varepsilon_{kj}), \quad \chi_k = \delta_k\theta(\theta^2 + 6\varepsilon_{ki}\varepsilon_{kj}) \tag{1.20}$$

$$\xi_k = \varepsilon_{ik}\varepsilon_{jk} + 9\delta_i\delta_j, \quad \eta_{ki} = \varepsilon_{ki}\varepsilon_{ij} + \theta\varepsilon_{ji} + 9\delta_k\varepsilon_{ik}$$

$$p_k = 4\lambda_k - \xi_k^2 + 108\delta_i^2\delta_j^2, \quad q_{ki} = 2\mu_{ki} - \eta_{ki}\xi_i + 108\delta_k^2\delta_j\varepsilon_{ik}$$

$$r_{ki} = 4\nu_{ki} - \eta_{ki}^2 - 2\eta_{kj}\xi_i + 108\delta_k^2(\varepsilon_{ik}^2 + 2\varepsilon_{jk}\delta_j)$$

$$s_k = 2\pi_k - \xi_i\xi_j - \eta_{ki}\eta_{kj} + 108\delta_k^2(\delta_i\delta_j + \varepsilon_{ik}\varepsilon_{jk})$$

$$\rho_{ki} = p_i r_{ki}^3 + 16 s_k q_{ki}^3 - (q_{ki}r_{ki} + 9p_i s_k)^2 + 108 p_i^2 s_k^2$$

$$\sigma_{ki} = r_{ki}r_{kj}^3 + 16 s_k^3 q_{kj} - (s_k r_{kj} + 9 q_{kj} r_{ki})^2 + 108 r_{ki}^2 q_{kj}^2$$

In this notation

$$D_{ki}(t) = p_j t^6 + 2q_{kj}t^5 + r_{kj}t^4 + 2s_k t^3 + r_{ki}t^2 + 2q_{ki}t + p_i \tag{1.21}$$

Theorem 2. Suppose that the following conditions are satisfied at a point $\upsilon \in V$:
1. inequalities (1.8) hold for any permutation i, j, k of the indices 1, 2, 3;
2. a permutation i, j, k of the numbers 1, 2, 3 exists such that at least one of the following combinations of conditions (1.22)–(1.30) holds

$$\beta_k\varepsilon_{ij} \geq 0, \quad \beta_k\varepsilon_{ji} \geq 0, \quad \beta_k\varepsilon_{ki} \geq 0, \quad \beta_k\varepsilon_{kj} \geq 0$$

$$\theta\,\mathrm{sign}(\beta_k) + 2\sqrt{\varepsilon_{ki}\varepsilon_{kj}} > 0, \quad \varepsilon_{ij}^2 + \varepsilon_{ji}^2 > 0 \tag{1.22}$$

$$p_i \geq 0, \quad p_j \geq 0, \quad r_{ki} \geq 0, \quad r_{kj} \geq 0, \quad s_k \geq 0, \quad \varphi_{ki} > 0, \quad \varphi_{kj} > 0 \tag{1.23}$$

$$p_i \geq 0, \quad p_j \geq 0, \quad q_{ki} \geq 0, \quad q_{kj} \geq 0, \quad r_{ki} \geq 0$$

$$r_{kj} \geq 0, \quad s_k + \sqrt{r_{ki}r_{kj}} > 0 \tag{1.24}$$

$$p_i \geq 0, \quad p_j \geq 0, \quad q_{kj} \geq 0, \quad r_{ki} \geq 0, \quad s_k \geq 0, \quad \tau_{kj} > 0, \quad \varphi_{ki} > 0 \tag{1.25}$$

$$p_i \geq 0, \quad p_j \geq 0, \quad q_{ki} \geq 0, \quad r_{kj} \geq 0, \quad s_k \geq 0, \quad \tau_{ki} > 0, \quad \varphi_{kj} > 0 \tag{1.26}$$

$$p_i \geq 0, \quad p_j \geq 0, \quad r_{kj} \geq 0, \quad s_k > 0, \quad \rho_{ki} > 0, \quad \varphi_{kj} \geq 0 \tag{1.27}$$

$$p_i \geq 0, \quad p_j \geq 0, \quad r_{ki} \geq 0, \quad s_k > 0, \quad \rho_{kj} > 0, \quad \varphi_{ki} \geq 0 \tag{1.28}$$

$$p_i \geq 0, \quad p_j \geq 0, \quad q_{ki} \geq 0, \quad q_{kj} > 0, \quad r_{ki} > 0, \quad \sigma_{ki} > 0 \tag{1.29}$$

$$p_i \geq 0, \quad p_j \geq 0, \quad q_{ki} > 0, \quad q_{kj} \geq 0, \quad r_{kj} > 0, \quad \sigma_{kj} > 0 \tag{1.30}$$

where

$$\varphi_{ki} = q_{ki} + \sqrt{p_i r_{ki}}, \quad \tau_{ki} = r_{ki} + 4\sqrt{q_{ki}s_k}$$

Then the ellipticity condition holds at a given point $\upsilon \in V$ of the material.

For the proof one need only compare conditions (1.22)–(1.30) with expressions (1.6) and (1.21) for $A_{ki}(t)$, $B_{ki}(t)$, $D_{ki}(t)$ and use Theorem 1.

2. If the point $\upsilon \in V$ under consideration is such that the numbers α_q, β_q (q = 1, 2, 3) are either all positive or all negative, one can derive an ellipticity criterion different from that of Theorem 1, in that it does not require an investigation of polynomials, but consists of a finite system of elementary inequalities. We shall first prove an auxiliary proposition.

Let Λ be the collection of real quadratic forms in three real variables x_1, x_2 and x_3. Let

$$L(\mathbf{x}) = \sum_{m,n=1}^{3} a_{mn}x_m x_n, \quad a_{mn} = a_{nm} \quad (m,n = 1,2,3)$$

$$\mathbf{x} = (x_1, x_2, x_3), \quad A = \| a_{mn} \| \quad (m,n = 1,2,3)$$

$$P = \{L \in \Lambda: L(\mathbf{x}) > 0, \quad \mathbf{x} \in R^3, \quad \mathbf{x} \neq 0\}$$

$$N = \{L \in \Lambda: L(\mathbf{x}) < 0, \quad \mathbf{x} \in R^3, \quad \mathbf{x} \neq 0\} \tag{2.1}$$

$$P^0 = \{L \in \Lambda: L(\mathbf{x}) \geqslant 0, \quad \mathbf{x} \in R^3\}$$

$$N^0 = \{L \in \Lambda: L(\mathbf{x}) \leqslant 0, \quad \mathbf{x} \in R^3\}$$

Lemma 2. For any form $L \in \Lambda$

$$(\det A \neq 0) \Leftrightarrow \{(L \in P) \vee (L \in N) \vee [(L \notin P^0) \wedge$$

$$\wedge (L \notin N^0)]\} \wedge [\forall S: L|_S \equiv 0] \tag{2.2}$$

where S is any plane in R^3 that passes through the origin O and $L|_S$ denotes the restriction if L to S.

Proof. Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be the natural basis of R^3. Let $\mathbf{L}$ denote the symmetric second-rank tensor over R^3 with component matrix A in the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. Then obviously

$$L(\mathbf{x}) = \mathbf{x} \cdot \mathbf{L} \cdot \mathbf{x} \tag{2.3}$$

We will now use the fact that any symmetric second-rank tensor over R^3 has at least one orthogonal triple of eigenvectors. Let l_1, l_2 and l_3 be an orthonormal trihedron of eigenvectors of $\mathbf{L}$, let L_1, L_2 and L_3 be the corresponding eigenvalues and let X_q be the components of a vector $\mathbf{x}$ relative to l_1, l_2 and l_3. Then it follows from (2.3) that

$$L(\mathbf{x}) = L_1 X_1^2 + L_2 X_2^2 + L_3 X_3^2 \tag{2.4}$$

Apply the logical operation of negation of the equivalence (2.2)

$$(\det A = 0) \Leftrightarrow [L \in (P^0 \setminus P)] \vee [L \in (N^0 \setminus N)] \vee [\exists S: L|_S \equiv 0] \tag{2.5}$$

Assuming that $\det A = 0$, we shall show that the left-hand side of the statement (2.5) implies the right-hand side. Indeed, since the determinant of A is an invariant of $\mathbf{L}$, it follows that $\det A = L_1 L_2 L_3$. Consequently, a permutation i, j, k of the numbers 1, 2, 3 exists such that $L_k = 0$. If at the same time $L_i \geqslant 0$, $L_j \geqslant 0$, it follows from (2.4) that $L \in P^0 \backslash P$ (see (2.1)). Similarly, if $L_i \leqslant 0$, $L_j \leqslant 0$ we get $L \in N^0 \backslash N$. But if $L_i > 0$, $L_j < 0$ or $L_i < 0$, $L_j > 0$, it follows from (2.4) that

$$L(\mathbf{x}) = \operatorname{sign}(L_i)(|L_i|^{1/2} X_i + |L_j|^{1/2} X_j)(|L_i|^{1/2} X_i - |L_j|^{1/2} X_j)$$

i.e. $L(\mathbf{x})$ vanishes identically in the planes $|L_i|^{1/2} X_i \pm |L_j|^{1/2} X_j = 0$, each of which passes through the origin O. Thus the right-hand implication of (2.5) is valid.

We will now prove the reverse implication, assuming that the right-hand side of (2.5) is valid. Omitting the trivial cases $L \in P^0 \backslash P$ and $L \in N^0 \backslash N$, let us assume that a plane S exists containing the point O such that $L|_S \equiv 0$. Choosing a suitable permutation i, j, k of 1, 2, 3, we can always write the equation of S as $X_k = \lambda X_i + \mu X_j$, where λ, μ are constants. We then deduce from (2.4) that

$$L|_S = (L_i + L_k \lambda^2) X_i^2 + (L_j + L_k \mu^2) X_j^2 + 2 L_k \lambda \mu X_i X_j$$

Since $L|_S = 0$, this implies the inequalities

$$L_i + L_k \lambda^2 = 0, \quad L_j + L_k \mu^2 = 0, \quad L_k \lambda \mu = 0$$

But it is obvious that these equalities are mutually compatible only if $L_1 L_2 L_3 = 0$, completing the proof of the lemma.

Relations (2.2) and (2.5) have a simple geometrical interpretation. Consider the second-order surface $\mathbf{x} \cdot \mathbf{L} \cdot \mathbf{x} = \pm c^2$ (c = const > 0), which may be associated with any self-adjoint linear transformation of the space R^3 and is otherwise known as the Cauchy quadric. Then formulae (2.5) and (2.2) mean that the Cauchy quadric for a singular transformation is a linear surface (in our case, either an elliptic cylinder, two conjugate hyperbolic cylinders or a pair of parallel planes); but for a non-singular transformation it is not linear and is reducible either to an ellipsoid or to a pair of conjugate hyperboloids (single- or double-sheeted).

Lemma 2, which relates the condition $\det A \neq 0$ to the behaviour of the form $L(\mathbf{x})$ both over the whole of the space R^3 and in arbitrary planes S through the origin O, is a major tool of the subsequent analysis. Consider the quadratic form $\Phi(\mathbf{M}, \mathbf{x})$ corresponding to the acoustic tensor $\mathbf{Q}(\mathbf{N})$ (the vector $\mathbf{M}$ whose components in the basis Σ are $M_q = \upsilon_q N_q$, $q = 1, 2, 3$, appears here as a parameter)

$$\Phi(\mathbf{M},\mathbf{x}) = J\mathbf{x}\cdot\mathbf{Q}(\mathbf{N})\cdot\mathbf{x} = \sum_{q=1}^{3} \sigma_q x_q^2 + 2(\omega_3 x_1 x_2 + \omega_1 x_2 x_3 + \omega_2 x_3 x_1) \tag{2.6}$$

$$\omega_k = \gamma_k M_i M_j, \quad \sigma_k = \alpha_j M_i^2 + \alpha_i M_j^2 + \beta_k M_k^2$$

$$(i,j,k) = (1,2,3), \quad (2,3,1), \quad (3,1,2)$$

Applying Lemma 2 to $\Phi(\mathbf{M}, \mathbf{x})$, we obtain the following.

Theorem 3. A homogeneous isotropic compressible elastic material has the ellipticity property at a given point $\upsilon \in V$ if and only if, for any non-zero vector $\mathbf{M}$, the following statement is true

$$\{(\Phi \in P) \vee (\Phi \in N) \vee [(\Phi \notin P^0) \wedge (\Phi \notin N^0)]\} \wedge [\forall S: \Phi|_S \equiv 0] \tag{2.7}$$

where S is an arbitrary plane in R^3 passing through the origin O and $\Phi|_S$ is the restriction of Φ to S.

Note that the condition $\Phi \in P$ is equivalent to strong ellipticity of the material, while $\Phi \in P^0$ means that the material satisfies Hadamard's inequality. The inclusions $\Phi \in N$, $\Phi \in N^0$ can be explained similarly, but applied to the potential Π with reversed sign. It can be shown that for the form $\Phi(\mathbf{M}, \mathbf{x})$ to belong (for all $\mathbf{M} \neq 0$) to the spaces P, N, P^0 and N^0, respectively, it is necessary and sufficient that (see, e.g. [15, 16])

$$\alpha_k > 0, \ \beta_k > 0, \ G_k^{\pm} > 0, \ [(\gamma_i^m < 0) \wedge (\gamma_j^n < 0)] \Rightarrow \zeta_k^{m,n} > 0 \tag{2.8}$$

$$\alpha_k < 0, \ \beta_k < 0, \ H_k^{\pm} < 0, \ [(\gamma_i^m > 0) \wedge (\gamma_j^n > 0)] \Rightarrow \zeta_k^{m,n} > 0 \tag{2.9}$$

$$\alpha_k \geqslant 0, \ \beta_k \geqslant 0, \ G_k^{\pm} \geqslant 0, \ [(\gamma_i^m < 0) \wedge (\gamma_j^n < 0)] \Rightarrow \zeta_k^{m,n} \geqslant 0 \tag{2.10}$$

$$\alpha_k \leqslant 0, \ \beta_k \leqslant 0, \ H_k^{\pm} \leqslant 0, \ [(\gamma_i^m > 0) \wedge (\gamma_j^n > 0)] \Rightarrow \zeta_k^{m,n} \geqslant 0 \tag{2.11}$$

$$G_k^{\pm} = \gamma_k^{\pm} + \sqrt{\beta_i \beta_j}, \quad H_k^{\pm} = \gamma_k^{\pm} - \sqrt{\beta_i \beta_j}$$

$$\zeta_k^{m,n} = \beta_k \gamma_k^{mn} - \gamma_i^m \gamma_j^n + [\beta_k \beta_j - (\gamma_i^m)^2]^{1/2} [\beta_k \beta_j - (\gamma_j^n)^2]^{1/2} \tag{2.12}$$

It is assumed here that the superscripts m and n take values plus or minus, with the product behaving in accordance with the multiplication rule for the numbers +1 and −1.

Theorem 4. If the point $\upsilon \in V$ under consideration is such that the parameters α_q and β_q are simultaneously positive or simultaneously negative, then a necessary and sufficient condition for a homogeneous isotropic compressible material to have the ellipticity property at $\upsilon \in V$ is (i, j, k denotes an arbitrary permutation of the indices 1, 2, 3)

$$\gamma_k^{\pm} \operatorname{sign}(\alpha_k) + \sqrt{\beta_i \beta_j} > 0 \tag{2.13}$$

$$(\Phi \in P) \vee (\Phi \in N) \vee [(\Phi \notin P^0) \wedge (\Phi \notin N^0)] \tag{2.14}$$

Proof. Necessity follows from Theorems 1 and 3, respectively. It remains to prove sufficiency. We will show that inequalities (2.13) imply

$$\forall S: \Phi|_S \not\equiv 0 \tag{2.15}$$

where S is a plane through the origin O. To do this, consider a plane S with equation $x_k = \lambda x_i + \mu x_j$ (λ, μ = const). It follows from (2.6) that

$$\Phi|_S = f_k x_i^2 + g_k x_j^2 + 2h_k x_i x_j$$

$$f_k = \sigma_k \lambda^2 + 2\omega_j \lambda + \sigma_i, \quad g_k = \sigma_k \mu^2 + 2\omega_i \mu + \sigma_j$$

$$h_k = \sigma_k \lambda\mu + \omega_i \lambda + \omega_j \mu + \omega_k$$

Since by assumption the parameters α_i, α_j, β_k are either all positive or all negative, it follows that $\sigma_k \neq 0$. Consequently, f_k is a quadratic trinomial in λ with non-zero leading coefficient. Its discriminant is

$$\Delta = \omega_j^2 - \sigma_k\sigma_i = -[\alpha_j\beta_i M_i^4 + \alpha_i\alpha_k M_j^4 + \alpha_j\beta_k M_k^4 + \\ +(\alpha_j\alpha_k + \alpha_i\beta_i)M_i^2 M_j^2 + (\alpha_i\alpha_j + \alpha_k\beta_k)M_j^2 M_k^2 + (\alpha_j^2 - \gamma_j^2 + \beta_i\beta_k)M_k^2 M_i^2] \tag{2.16}$$

The bracketed expression on the right of (2.16) is a quadratic form in M_i^2, M_j^2, M_k^2, all of whose coefficients, with the possible exception of the last, are positive by assumption. Since $\gamma_j^+ + \gamma_j^- = 2\alpha_j$, $\gamma_j^+ - \gamma_j = 2\gamma_j$, as follows from (1.3), we deduce from (2.13) that

$$\alpha_j^2 - \gamma_j^2 + \beta_i\beta_k + 2|\alpha_i|\sqrt{\beta_i\beta_k} = [\gamma_j^+ \operatorname{sign}(\alpha_j) + \sqrt{\beta_i\beta_k}] \times [\gamma_j^- \operatorname{sign}(\alpha_j) + \sqrt{\beta_i\beta_k}] > 0$$

Consequently, $\Delta < 0$ for any vector $\mathbf{M}$ other than zero. Therefore f_k is either positive or negative for all $\lambda \in R$ and all $\mathbf{M}$ other than zero. In any case, the restriction $\Phi|_S$ is not identically zero. And since, by suitable choice of a permutation i, j, k of 1, 2, 3 we can write the equation of any plane S through the origin O as $x_k = \lambda x_i + \mu x_j$, the validity of condition (2.15) is thus established.

Thus, conditions (2.13) and (2.14) imply condition (2.7). But then it follows from Theorem 3 that the material is elliptic. This proves the theorem.

Corollaries to Theorems 3 *and* 4.

1. If the given point $\upsilon \in V$ is such that the parameters α_q, β_q (q = 1, 2, 3) are either simultaneously positive or simultaneously negative, then a sufficient condition for ellipticity of the material at the point considered is

$$\gamma_k^{\pm} \operatorname{sign}(\alpha_k) + \sqrt{\beta_i\beta_j} > 0$$

$$\{[\gamma_i^m \operatorname{sign}(\alpha_k) < 0] \wedge [\gamma_j^n \operatorname{sign}(\alpha_k) < 0]\} \Rightarrow \zeta_k^{m,n} \neq 0$$

where i, j, k is an arbitrary permutation of the indices 1, 2, 3, and m, n is any combination of the signs plus/minus, and the quantities $\zeta_k^{m,n}$ are as defined by (2.12).

2. Suppose that the following inequalities hold in the space of principal extensions

$$\alpha_k > 0, \quad \beta_k > 0, \quad \gamma_k^{\pm} + \sqrt{\beta_i\beta_j} > 0$$

Then a necessary and sufficient condition for the material to be elliptic at each point $\upsilon \in V$ is that its domain of strong ellipticity should coincide with the domain in which Hadamard's inequality is satisfied.

3. If at each point $\upsilon \in V$

$$\alpha_k > 0, \quad \beta_k > 0$$

$$[(\gamma_i^m < 0) \wedge (\gamma_j^n < 0)] \Rightarrow \beta_k\gamma_k^{mn} - \gamma_i^m\gamma_j^n \geqslant 0$$

where i, j, k is an arbitrary permutation of the indices 1, 2, 3 and m, n is any combination of signs plus/minus, then the domains of ellipticity and strong ellipticity of the material coincide.

The proofs of these corollaries are omitted.

To incorporate various physical considerations relating to the behaviour of elastic materials under deformation, different restrictions have been proposed, formulated as inequalities imposed on the constitutive relations of the elastic bodies, which they call constitutive or supplementary inequalities. Such, for example, are the Baker–Ericksen inequalities [1, 2, 17]

$$(t_i - t_j)/(\upsilon_i - \upsilon_j) > 0, \quad \upsilon_i \neq \upsilon_j \tag{2.17}$$

where t_q ($q = 1, 2, 3$) are the principal stresses, i.e. the eigenvalues of the Cauchy stress tensor, which are simply the principal extensions in the case of an isotropic material. By the strong version of the Baker–Ericksen inequalities we shall mean, in addition to the condition that quotients of the form $(t_i - t_j)/(\upsilon_i - \upsilon_j)$ should be positive for $\upsilon_i \neq \upsilon_j$, that these quotients have positive limits at $\upsilon_i \to \upsilon_j$.

The conditions $\partial t_q/\partial\upsilon_q > 0$ (no summation over q) are known as the TE^+-inequalities [1, 2].

Some constitutive inequalities are formulated in differential form, as the condition that the contraction of the strain rate tensor with some frame-indifferent derivative with respect to time of the Cauchy stress tensor should be positive. Examples are the Coleman–Noll inequalities [1, 2, 18]

$$[\mathbf{S}(\mathbf{C}, \boldsymbol{\epsilon}) + \mathbf{T} \operatorname{tr} \boldsymbol{\epsilon} - \frac{1}{2}(\boldsymbol{\epsilon} \cdot \mathbf{T} + \mathbf{T} \cdot \boldsymbol{\epsilon})] \cdot\cdot\ \boldsymbol{\epsilon} > 0 \quad (\boldsymbol{\epsilon} \neq 0) \tag{2.18}$$

Hill's inequality [19]

$$[\mathbf{S}(\mathbf{C}, \boldsymbol{\epsilon}) + \mathbf{T} \operatorname{tr} \boldsymbol{\epsilon}] \cdot\cdot\ \boldsymbol{\epsilon} > 0 \quad (\boldsymbol{\epsilon} \neq 0) \tag{2.19}$$

and the hydrostatic stability condition [20]

$$\mathbf{S}(\mathbf{C}, \boldsymbol{\epsilon}) \cdot\cdot\ \boldsymbol{\epsilon} > 0 \quad (\boldsymbol{\epsilon} \neq 0) \tag{2.20}$$

where $\mathbf{T}$ is the Cauchy stress tensor, $\boldsymbol{\epsilon}$ is the strain rate tensor and $\mathbf{S}$ is the derivative of the stress tensor in Jaumann's sense. In an isotropic material, the dependence of $\mathbf{S}$ on the deformation gradient reduces to its dependence on the Finger strain tensor, and the function $\mathbf{S}(\mathbf{F}, \boldsymbol{\epsilon})$ is easily calculated, given the potential $\Pi(\upsilon_1, \upsilon_2, \upsilon_3)$.

Note that the condition for the longitudinal elasticity to be positive may be expressed in terms of the tensor $\mathbf{S}$, as follows:

$$\mathbf{S}(\mathbf{C}, \mathbf{xx}) \cdot\cdot\ \mathbf{xx} > 0 \quad (\mathbf{x} \neq 0) \tag{2.21}$$

where $\mathbf{x}$ is any vector.

Checking for ellipticity becomes easier if the elastic material is required to satisfy a constitutive inequality. More precisely the following holds.

Theorem 5. Assume that at a point $\upsilon \in V$ at least one of the following conditions 1–4 holds

1. the Baker–Ericksen inequalities (2.17) hold (strong version) and the material either satisfies the TE^+-condition, the condition for its longitudinal elasticity to be positive (2.21), or the Coleman–Noll inequality (2.18);
2. The TE^+-condition and Hill's inequality (2.19) hold;
3. the material is hydrostatically stable (2.20);
4. the acoustic numbers for the principal longitudinal and principal transverse amplitudes are either all positive or all negative.

Then conditions (2.13) and (2.14) are necessary and sufficient for the material to be elliptic at $\upsilon \in V$.

The proof, which uses Theorems 3 and 4, is omitted.

Theorem 5 states that, subject to physically reasonable assumptions, the ellipticity criterion for a compressible material comprises a finite sequence of elementary inequalities involving no subsidiary parameters; there is no need to appeal to the general Theorem 1, which requires investigation of a sixth-degree polynomial. The composite condition (2.14) is easily interpreted using relations (2.8)–(2.11).

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